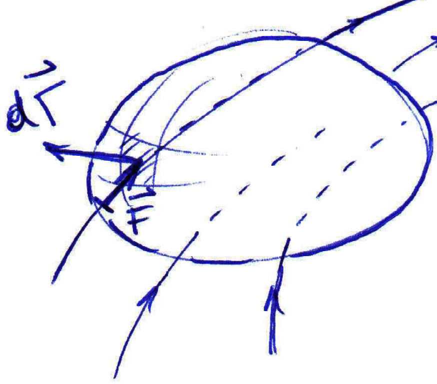
WFS: Herleitung des Kirchhoff-Helmholtz Integrals

Algorithmen in Akustik und Computermusik 2, SE+UE. IEM, Franz Zotter, 2008.

1) Gaußscher Integralsatz

a) Quellenstärke eines Vektorfeldes $\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$ als Hüllenintegral über differentiellen Fluß dQ.

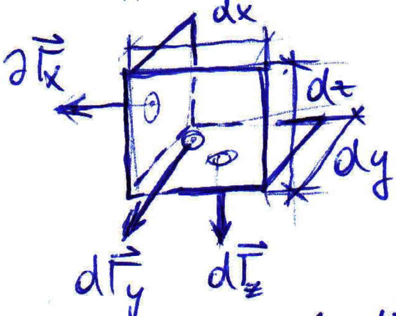
$$Q = \int_{\Omega} dQ$$



$$dQ = \vec{F} \cdot d\vec{\Gamma}$$

$$\Rightarrow Q = \oint_{\partial\Omega} \vec{F} \cdot d\vec{\Gamma}$$

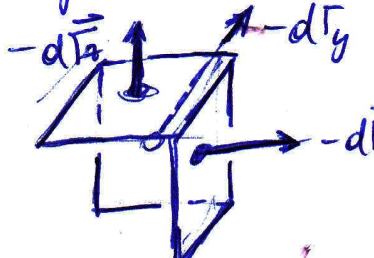
b) Quellenstärke als Volumsintegral über differentielle Hüllen ...



$$d\vec{\Gamma}_x = -dy\,dz \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad dQ_x = -dy\,dz\,F_x$$

$$d\vec{\Gamma}_y = -dx\,dz \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad dQ_y = -dx\,dz\,F_y$$

$$d\vec{\Gamma}_z = -dx\,dy \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad dQ_z = -dx\,dy\,F_z$$



$$F_{x+dx} = F_x + \frac{\partial F_x}{\partial x} dx \qquad dQ_{x+dx} = dy\,dz\,F_x + \frac{\partial F_x}{\partial x} dx\,dy\,dz$$

$$F_{y+dy} = F_y + \frac{\partial F_y}{\partial y} dy \qquad dQ_{y+dy} = dx\,dz\,F_y + \frac{\partial F_y}{\partial y} dy\,dx\,dz$$

$$F_{z+dz} = F_z + \frac{\partial F_z}{\partial z} dz \qquad dQ_{z+dz} = dx\,dy\,F_z + \frac{\partial F_z}{\partial z} dz\,dx\,dy$$

$$\begin{aligned} dQ &= dQ_x + dQ_y + dQ_z + dQ_{x+dy} + dQ_{y+dy} + dQ_{z+dz} \\ &= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx\,dy\,dz \\ &= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \vec{F} \cdot dV = \nabla \cdot \vec{F}\, dV \end{aligned}$$

$$\Rightarrow Q = \oint_{\Omega} \nabla \cdot \vec{F}\, dV$$

$$\Rightarrow \boxed{\oint_{\partial\Omega} \vec{F} \cdot d\vec{\Gamma} = \oint_{\Omega} \nabla \cdot \vec{F}\, dV}$$

2) 2. Greensche Identität, setze

$$\vec{F} = g\nabla p - p\nabla g$$ in Gaußschen Integralsatz ein

$$\underline{\underline{\oint_{\partial\Omega} (g\nabla p - p\nabla g)\cdot d\vec{\Gamma}}} = \oint_{\Omega} \nabla(g\nabla p - p\nabla g)\, dV$$

$$= \int_{\Omega} (\cancel{\nabla g\nabla p} + g\nabla^2 p - \cancel{\nabla p\nabla g} - p\nabla^2 g)\, dV$$

$$= \underline{\underline{\int_{\Omega} (g\nabla^2 p - p\nabla^2 g)\, dV}}$$

3. Greensche Identität,

Kirchhoff-Helmholtz-Integral:

setze ... $(\nabla^2 + k^2)p = 0 \Rightarrow \nabla^2 p = -k^2 p$ ... homogenes Schallfeld (Helmholtz-Gleichung)

und ... $(\nabla^2 + k^2) g = -\delta(\vec{r} - \vec{r}_0) \Rightarrow \nabla^2 g = -k^2 g - \delta(\vec{r}-\vec{r}_0)$

... Greensche Funktion (inhomogenes Schallfeld, Quellenfeld d. Punktquelle)

rechte Seite d. 2. Greenschen Identität:

$$\int_{\Omega} (g\nabla^2 p - p\nabla^2 g)\, dV = \int_{\Omega} (\cancel{-gk^2 p + pk^2 g} + p\,\delta(\vec{r}-\vec{r}_0))\, dV$$

$$= p$$

$$\Rightarrow \boxed{\begin{aligned} p &= \oint_{\partial\Omega} (g\nabla p - p\nabla g)\cdot d\vec{\Gamma} \\ &= \underline{\underline{\oint_{\partial\Omega} \left(g\frac{\partial}{\partial\vec{n}}p - p\frac{\partial}{\partial\vec{n}}g\right)\cdot d\Gamma}} \end{aligned}}$$

$\frac{\partial}{\partial\vec{n}}g$ ... Dipolquelle normal auf $\partial\Omega$

$\frac{\partial}{\partial\vec{n}}p$ ... Normalschnelle d. Schallfeldes