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WARPING OF 3D AMBISONIC RECORDINGS

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Abstract: *In order to modify or adapt the spatial image of higher order Ambisonic recordings, e.g. made with spherical microphone arrays, there is still a need for algorithms. Originally, warping of the recording image has been proposed, which is able to widen the spatial mapping on one side of the surround image, while squeezing the recorded scene on the opposite side. This so-called dominance effect has been proposed in the papers of Gerzon and can be regarded as useful sound engineering effect. Similar effects are only available for MS or double MS stereophony to adjust stereophonic coverage angle. The available literature about this warping does not present algorithms for higher order Ambisonics. This article investigates suitable algorithms, and intends to investigate new algorithms based on recurrence relations.*

Key words: Ambisonic warping, dominance effect, bilinear transform

1 INTRODUCTION

Ambisonic recording techniques have the advantage of a uniform surround resolution. As there are new higher-order microphone arrays for recording, the question of spatial manipulations becomes more and more relevant. This is particularly true, because the uniform surround resolution does not naturally support modification of the recording coverage angles, as common in stereophony. In principle, the transform on the ambisonic channels is only a matrix operation, however, either a precise knowledge of the involved polynomials is necessary, or the matrix has to be determined numerically.

In fact, the so-called "dominance" transform is known for 1st order Ambisonics, but the desirable generalization of this algorithm to higher orders is still missing. Essentially, the regarded transform is a bilinear transform along one spatial direction. In this direction, the angular mapping is be widened, while on the opposite side the acoustical image is compressed. For recording technicians the, warping allows to adjust the recording angle in which, e.g., orchestra instruments appear in their post-processing.

Why polar warping is sufficient. To generalize this type of transform to higher-order Ambisonics, it is most comfortable to use the z-axis as a warping direction, mathematically. Combined with a pre and post rotation, warping with regard to any direction becomes accessible: the surround field represented in Ambisonics is first rotated so that the respective direction of warping points upwards towards z, where it is warped to or from, and rotated back afterwards. As it becomes obvious later, this largely limits the description of warping to the associated Legendre functions that describe the dependency on the zenith angle, i.e. z-axis. These functions depend on the transformed polar angle, only, and their argument undergoes bilinear transformation.

Hence, the main issue in this contribution is to discuss the warping of the zenith angle, in order to strech the mapping of an Ambisonic signal away from the equator. This enlarges a particular region of the coverage angle, allowing for modification of the surround recording image.

With this comfortable focus on the Legendre functions, properties of suitable warping algorithms can be described in two different ways. Both eventually intend to yield re-expansion coefficients that are applied as a time/frequency independent signal matrix, which re-expands and warps given Ambisonic signals.

On the one hand, a numerical type algorithm samples and re-expands the warped associated Legendre functions into the un-warped Legendre functions. Herewith, the conversion coefficients are directly found, numerically. On the other hand, an analytic algorithm is desirable and could make use of the recurrence relations of the Legendre functions. The analytic approach is outlined in the paper, mainly to show its mathematical ingredients. It is not prepared yet to give details about the best and easiest implementation. Nevertheless, the numerical algorithm is fully functional and readily provides the desired coefficients at an easy implementation level, however at a lower level of mathematical elegance.

To avoid an increased loudness of the angularly stretched audio scene in contrast to the compressed part, an optional direction-dependent amplitude function is described, for which an analytic algorithm has been found.

Un-normalized spherical harmonics are used in this paper to keep the math simple:

$$Y_n^m(\varphi,\vartheta) = P_n^m(\cos(\vartheta))\, e^{\mathrm{i}m\varphi}. \tag{1}$$

Herein, $P_n^m(\cos(\vartheta))$ are the associate Legendre-functions, ϑ is the polar (or zenith) angle, and φ is the azimuth angle.

2 WARPING OF THE POLAR ANGLE

Given an Ambisonic signal $\phi_n^m(t)$ to linearly combine the spherical harmonics, the signal is expanded into continuous angles φ, ϑ, i.e. azimuth and zenith, repsectively,

$$f(\varphi,\vartheta,t) = \sum_{n=0}^{\mathrm{N}} \sum_{m=-n}^{n} Y_n^m(\varphi,\vartheta)\, \phi_n^m(t), \tag{2}$$

the task of an angular warping is to find the Ambisonic signal $\tilde{\phi}_n^m(t)$ that exhibits a warped polar (zenith) angle $\tilde{\vartheta}$

$$f(\varphi,\tilde{\vartheta},t) = \sum_{n=0}^{\tilde{\mathrm{N}}} \sum_{m=-n}^{n} Y_n^m(\varphi,\vartheta)\, \tilde{\phi}_n^m(t). \tag{3}$$

In both equations, $-n \le m \le n$, and $0 \le n \le \mathrm{N}$, or for the second equation with another limit $0 \le n \le \tilde{\mathrm{N}}$. The warped distribution in Eq. (3) can be expressed equivalently as a sum of warped spherical harmonics weighted by the original Ambisonics signals,

$$\sum_{n=0}^{\tilde{\mathrm{N}}} \sum_{m=-n}^{n} Y_n^m(\varphi,\vartheta)\, \tilde{\phi}_n^m = \sum_{n=0}^{\mathrm{N}} \sum_{m=-n}^{n} Y_n^m(\varphi,\tilde{\vartheta})\, \phi_n^m. \tag{4}$$

By integrating over the azimuth harmonics $e^{\mathrm{i}m'\varphi}$, orthogonality $\int e^{\mathrm{i}m'\varphi} e^{m\mathrm{i}\varphi}\, d\varphi = 2\pi\delta^{m'm}$ yielding a Kronecker delta can be used to reduce one summation

$$\sum_{n=0}^{\mathrm{N}} P_n^m(\mu)\, \tilde{\phi}_n^m = \sum_{n=0}^{\mathrm{N}} P_n^m(\tilde{\mu})\, \phi_n^m, \tag{5}$$

whereby $\mu = \cos(\vartheta)$, $\tilde{\mu} = \cos(\tilde{\vartheta})$ is used in the following for notational convenience. The search of re-expansion coefficients can be focused on the Legendre-functions at first.

Bilinear transform. In particular, the warping relation under consideration is a bilinear transform between normalized z-axes $\tilde{\mu}$ and μ, which always maps the points $\mu = \pm 1$ to $\tilde{\mu} \pm 1$ and remains monotony, however distorts all angles between. For instance, the equatorial point $\mu = 0$ is mapped to $\tilde{\mu} = \alpha$ by the angle $\epsilon = \arcsin(\alpha)$, c.f. Figure 1. This transform was proposed in the work of Gerzon [1] and also investigated in Sontacchi's thesis [2]:

$$\tilde{\mu} = \frac{\mu + \alpha}{1 + \alpha\mu}. \tag{6}$$

Re-expansion. The orthogonality relation of the associated Legendre functions, $\int_{-1}^{1} P_n^m(\mu)\, P_{n'}^m(\mu)\, d\mu = \frac{2(n-m)!}{(n+m)!(2n+1)} \delta_{n'n}$, can be used to reduce the sum on the

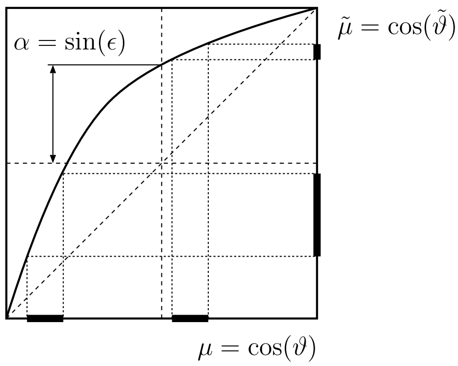


Figure 1: Mapping by bilinear transform of μ.

left side of Eq. (5) to a single term; integrating the Eq. (5) over $\frac{(n+m)!(2n+1)}{2(n-m)!} P_{n'}^m(\mu)\, d\mu \int$ yields

$$\tilde{\phi}_{n'}^m = \sum_n w_{n'n}^m\, \phi_n^m, \tag{7}$$

whereby the re-expansion coefficients are

$$w_{n'n}^m = \tfrac{(n+m)!(2n+1)}{2(n-m)!} \int_{-1}^{1} P_n^m(\tilde{\mu})\, P_{n'}^m(\mu)\, d\mu. \tag{8}$$

Inserting Eq. (7) into Eq. (5) shows that the coefficients $w_{n'n}^m$ also connect the original and the warped associated Legendre-functions,

$$\sum_{n'} P_{n'}^m(\mu) \sum_n w_{n'n}^m\, \phi_n^m = \sum_n P_n^m(\tilde{\mu})\, \phi_n^m$$
$$\Rightarrow \sum_{n'} w_{n'n}^m P_{n'}^m(\mu) = P_n^m(\tilde{\mu}). \tag{9}$$

3 NUMERICAL RE-EXPANSION

The straightforward way of determining $w_{n'n}^{m'}$ uses numerical integration of both the original and warped associated Legendre-functions.

Equally spaced discretization. The coefficients can be found numerically by discretizing their definition integral into L equally spaced points $\mu_l = -1 + l\,\frac{2}{\mathrm{L}-1}$ yielding $\tilde{\mu}_l = \frac{\mu_l + \alpha}{1 + \alpha\mu_l}$, and

$$w_{n'n}^m \approx \tfrac{(n'+m)!(2n'+1)}{2(n'-m)!} \frac{2}{\mathrm{L}} \sum_l P_n^m(\mu_l)\; P_{n'}^m(\tilde{\mu}_l). \tag{10}$$

In order to obtain sufficiently accurate values, the discretization number should be large compared to the resulting order $\tilde{\mathrm{N}}$ after warping, and error below -100dB could only be obtained using $\mathrm{L} = 10^5$ in own simulations.

Gauß-Legendre quadrature and discrete transform by inversion. Alternatively, either the Gauß-Legendre quadrature rule [3] can be used to obtain accurate results with much less points, the integration can be done by pseudo-inversion of the the sampled Legedre-functions

$$\begin{pmatrix} P_n^m(\tilde{\mu}_1) \\ \vdots \\ P_n^m(\tilde{\mu}_\mathrm{L}) \end{pmatrix} = \begin{pmatrix} P_m^m(\mu_1) & \cdots & P_{\tilde{\mathrm{N}}}^m(\mu_1) \\ \vdots & \ddots & \vdots \\ P_m^m(\mu_\mathrm{L}) & \cdots & P_{\tilde{\mathrm{N}}}^m(\mu_\mathrm{L}) \end{pmatrix} \begin{pmatrix} w_{m,m}^m \\ \vdots \\ w_{\tilde{\mathrm{N}},m}^m \end{pmatrix}$$

$$\tilde{\boldsymbol{p}}_{nm} = \boldsymbol{P}_m \, \boldsymbol{w}_{nm} \quad \Rightarrow \quad \boldsymbol{w}_{nm} = \boldsymbol{P}_m^\dagger \, \tilde{\boldsymbol{p}}_{nm}. \tag{11}$$

to find their weights, which was the favored option, here. For real-time calculation, pre-computed pseudo-inverse matrices $\boldsymbol{P}_m^\dagger$ can be used with a moderate number of points, e.g. $\mathrm{L} = 50$. However, as always, care must be taken to avoid spatial aliasing as higher orders components are generated by warping; theoretically, the order needs to be infinite to be entirely accurate.

Required re-expansion order $\tilde{\mathrm{N}}$. Table uses the discrete transform by inversion to retrieve, which maximum re-expansion order $\tilde{\mathrm{N}}$ is required for varying warping parameter $\alpha = \sin(\epsilon)$ and the original order $\mathrm{N} = 3$.

$\tilde{\mathrm{N}}(m, \epsilon)$	$0°$	$5°$	$10°$	$15°$	$20°$	$25°$	$30°$
$m = 0$	3	5	5	6	6	7	8
$m = 1$	3	5	5	6	6	7	7
$m = 2$	3	5	5	6	6	7	7
$m = 3$	3	4	5	5	6	6	7

Table 1: Required order $\tilde{\mathrm{N}}$ for warping an $\mathrm{N} = 3$ order Ambisonic signal by warping angles $\epsilon = \arcsin(\alpha)$ is shown, depending on the index m of the Legendre-functions. As a criterion, the re-expansion coefficients for normalized functions below $\frac{N_n^m}{N_{n'}^m} \, |w_{n'n}^m| > -30dB$ were omitted for all n.

4 ANALYTIC RE-EXPANSION

The integral in Eq. (8) re-expanding warped Legendre-functions is easily evaluated for the expressions in Fig. 2a,

$$w_{n'0}^0 = \tfrac{2n'+1}{2} \int_{-1}^{1} P_{n'}^0(\mu) \, d\mu = \delta_{n',0} \, , \tag{12}$$

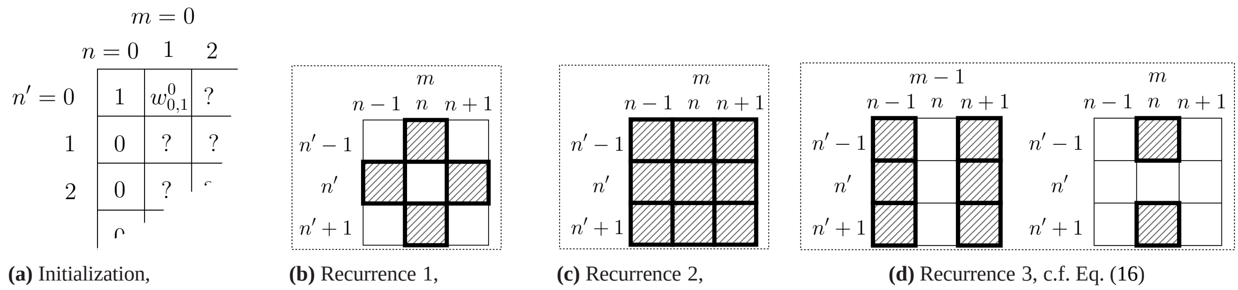
$$\begin{aligned} w_{0,1}^0 &= \tfrac{1}{2} \int_{-1}^{1} \tfrac{\mu+\alpha}{1+\alpha\mu} \, d\mu \\ &= \tfrac{\alpha^2-1}{2\alpha} \left(\ln(1+\alpha) - \ln(1-\alpha) \right) + \tfrac{1}{\alpha} \, . \end{aligned} \tag{13}$$

With the recurrence relations of the Legendre-functions, c.f. [4, 5], new recurrences are found for the re-expansion coefficients $w_{n'n}^{m'}$ of warping, c.f. App. A. Relations of the recurrences are shown in Fig. 2 and detailed below

$$\tfrac{(n+1)(n+m)}{2n+1} w_{n',n-1}^m - \tfrac{n(n-m+1)}{2n+1} w_{n',n+1}^m = \tfrac{(n'+2)(n'+m+1)}{2n'+3} w_{n'+1,n}^m - \tfrac{(n'-1)(n'-m)}{2n'-1} w_{n'-1,n}^m, \tag{14}$$

$$\begin{aligned} &\alpha \tfrac{(n+m)(n'+m+1)}{(2n+1)(2n'+3)} w_{n'+1,n-1}^m + \tfrac{n+m}{2n+1} w_{n',n-1}^m + \alpha \tfrac{(n'-m)(n+m)}{(2n+1)(2n'-1)} w_{n'-1,n-1}^m \\ &\quad + \alpha \tfrac{(n-m+1)(n'+m+1)}{(2n+1)(2n'+3)} w_{n'+1,n+1}^m + \tfrac{n-m+1}{2n+1} w_{n',n+1}^m + \alpha \tfrac{(n-m+1)(n'-m)}{(2n+1)(2n'-1)} w_{n'-1,n+1}^m = \\ &\qquad \tfrac{n'+m+1}{2n'+3} w_{n'+1,n}^m + \alpha \, w_{n',n}^m + \tfrac{n'-m}{2n'-1} w_{n'-1,n}^m, \end{aligned} \tag{15}$$

$$\begin{aligned} &\alpha \tfrac{(n-m+1)(n-m+2)(n'+m)}{(2n+1)(2n'+3)} w_{n'+1,n+1}^{m-1} + \tfrac{(n-m+1)(n-m+2)}{2n+1} w_{n',n+1}^{m-1} + \alpha \tfrac{(n-m+1)(n-m+2)(n'-m+1)}{(2n+1)(2n'-1)} w_{n'-1,n+1}^{m-1} \\ &\quad - \alpha \tfrac{(n+m-1)(n+m)(n'+m)}{(2n+1)(2n'+3)} w_{n'+1,n-1}^{m-1} - \tfrac{(n+m-1)(n+m)}{(2n+1)} w_{n',n-1}^{m-1} - \alpha \tfrac{(n+m-1)(n+m)(n'-m+1)}{(2n+1)(2n'-1)} w_{n'-1,n-1}^{m-1} = \\ &\qquad \sqrt{1-\alpha^2} \tfrac{(n'-m)(n'-m+1)}{2n'-1} w_{n'-1,n}^m - \sqrt{1-\alpha^2} \tfrac{(n'+m)(n'+m+1)}{2n'+3} w_{n'+1,n}^m. \end{aligned} \tag{16}$$



(a) Initialization, c.f. Eqs. (12)(13)

(b) Recurrence 1, c.f. Eq. (14)

(c) Recurrence 2, c.f. Eq. (15)

(d) Recurrence 3, c.f. Eq. (16)

Figure 2: Recurrence theorems for warped Legendre functions.

5 MAGNITUDE EMPHASIS

The angular warping enlarges sources in the stretched region while keeping the same source amplitude at the direction of the source. By the enlarging or squeezing of sources with finite extent, the energy, i.e. loudness, of the affected content might change. It is therefore useful to regard the mapping relation to find a compensation for this effect. Also apart from this, such an angular loudness modifier might be beneficial.

The enlargement σ of sources can be described by the derivative of the mapping relation

$$\sigma = \frac{\partial\tilde{\mu}}{\partial\mu} = \frac{1}{\frac{\partial\mu}{\partial\tilde{\mu}}} = \frac{1-\alpha^2}{(1+\alpha\mu)^2} = \frac{(1-\alpha\tilde{\mu})^2}{1-\alpha^2}. \tag{17}$$

Consequently, to obtain an energy-preserving warping operation, regions with increased sources must be weighted accordingly. The de-emphasis after warping is

$$\frac{1}{\sqrt{\sigma}} = \frac{\sqrt{1-\alpha^2}}{1-\alpha\tilde{\mu}}, \tag{18}$$

or, alternatively, the pre-emphasis before warping

$$\frac{1}{\sqrt{\sigma}} = \frac{1+\alpha\mu}{\sqrt{1-\alpha^2}}. \tag{19}$$

Clearly, the second variant is easier to implement as it is only a multiplication by a first order polynomial, for which there exists a recurrence relation of the Legendre functions.
First order magnitude emphasis. Even without There are already high level algorithms as sketched in [6](Referenz Student+Svenson, AES London, 2011), which use modified sets of basis functions to attenuate particular directions; similarly, also the paper [7] allows to suppress signals from specified angular regions in the surround image. The presented algorithm here shall be simpler but also relies on a general concept.

Given the Ambisonic signal $\phi_n^m(t)$ as expanded in Eq. (2), the task of an angular loudness modification is to find the Ambisonic signals $\tilde{\phi}_n^m(t)$ that represent the original Ambisonic signal multiplied by an angular gain function

$$f(\varphi,\vartheta,t)\cdot g(\varphi,\vartheta) = \sum_{n=0}^{\mathrm{N}}\sum_{m=-n}^{n} Y_n^m(\varphi,\vartheta)\,\tilde{\phi}_n^m(t). \tag{20}$$

In general, this can be achieved by the Gaunt coefficients [8](Referenz, Driscoll Healy) when given the spherical harmonics decomposition $\gamma_{n'}^{m'}$ of the gain function. With the Gaunt/Clebsh-Gordan coefficients, the modified Ambisonic signals can be obtained $\tilde{\phi}_n^m(t) = \sum_{n',n'',m',m''} C_{n'',n',n}^{m'',m',m}\gamma_{n'}^{m'}\,\phi_{n''}^{m''}(t)$.

This section presents a simpler algorithm, which multiplies the audio scene with a rotationally symmetric cardioid amplitude pattern. Given algorithms to rotate the Ambisonic sound field with rotation matrices[1], in fact, it is sufficient to describe multiplications of the Ambisonic field with a cardioid amplitude pattern pointing upwards, i.e., with the zenith angle ϑ, a continuous, normalized gain function

$$g(\vartheta) = \frac{1+\alpha\cos(\vartheta)}{\sqrt{1+\alpha^2}}. \tag{21}$$

Essentially, it is clear that a multiplication of an Ambisonic signal with this rotationally symmetric pattern will only affect the shape with respect to the zenith angle, while the shape in the azimuth is retained. Therefore, the desired transformation only depends on relation between the zenith angle related Legendre functions. Omitting their normalization for simplicity, this is expressed as:

$$\frac{1+\alpha\mu}{\sqrt{1+\alpha^2}}\sum_n P_n^m(\mu)\,\phi_n^m = \sum_{n''} P_{n''}^m(\mu)\,\tilde{\phi}_{n''}^m. \tag{22}$$

The relation between the coefficients ϕ_n^m and $\tilde{\phi}_n^m$ corresponds to

$$\begin{aligned}\tilde{\phi}_{n'}^m &= \sum_{n=0}^{\mathrm{N}} \phi_n^m \tfrac{(n'+m)!(2n'+1)}{2(n'-m)!}\int_{-1}^{1}\frac{1+\alpha\mu}{\sqrt{1+\alpha^2}}\,P_n^m(\mu)\,P_{n'}^m(\mu)\,d\mu\\ &= \sum_{n=0}^{\mathrm{N}} g_{n'n}^m\,\phi_n^m.\end{aligned} \tag{23}$$

is derived in the appendix by utilizing the recurrence and orthogonality of the Legendre functions.

Loudness modification for Ambisonic signals. As given in Eq. (47), the loudness modified Ambisonic singals $\tilde{\phi}_n^m(t)$ are obtained from

$$\tilde{\phi}_n^m(t) = \frac{\alpha\,\frac{n+m}{2n+1}\,\phi_{n+1}^m(t) + \phi_n^m(t) + \alpha\,\frac{n-m+1}{2n+1}\,\phi_{n-1}^m(t)}{\sqrt{1+\alpha^2}}. \tag{24}$$

In general, this will yield a signal of the order $\mathrm{N}+1$ when the given signal was of the order N. Note that the inverse weighting operation with $(1+\alpha\mu)^{-1}$ is not so simple, however, there is a solution in the appendix, for some special case with a priori knowledge.

6 CONCLUSION

Numerical and analytical relations for warping were shown and a detailed theoretical description was given. Based on this theoretical considerations, a practical application of warping to higher order Ambisonics is now feasible. Such a spatial transformation will be a useful tool for editing 3D higher order Ambisonics recordings.

7 ACKNOWLEDGMENTS

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A RECURRENCES FOR WARPING

The relation with unknown re-expansion

$$P_n^m(\tilde{\mu}) = \sum_{n'',m''} w_{n''n}^{m''m} P_{n''}^{m''}(\mu), \tag{25}$$

can be modified by inserting three different recurrence relations

$$(1-\mu^2)\frac{d}{d\mu} P_n^m(\mu) = \frac{(n+1)(n+m)}{(2n+1)} P_{n-1}^m(\mu) - \frac{n(n-m+1)}{(2n+1)} P_{n+1}^m(\mu), \tag{26}$$

$$\mu P_n^m(\mu) = \frac{n+m}{2n+1} P_{n-1}^m(\mu) + \frac{n-m+1}{2n+1} P_{n+1}^m(\mu), \tag{27}$$

$$\sqrt{1-\mu^2} P_n^m(\mu) = \frac{(n-m+1)(n-m+2)}{2n+1} P_{n+1}^{m-1}(\mu) - \frac{(n+m-1)(n+m)}{2n+1} P_{n-1}^{m-1}(\mu). \tag{28}$$

Recurrence 1. The application of the chain rule is necessary to employ the first recurrence Eq.(26)

$$\frac{d}{d\mu} P_n^m(\tilde{\mu}) = \sum_{n''} w_{n''n}^m \frac{d}{d\mu} P_{n''}^m(\mu), \tag{29}$$

$$\frac{(1-\alpha\tilde{\mu})^2}{1-\alpha^2} \frac{d}{d\tilde{\mu}} P_n^m(\tilde{\mu}) = \sum_{n''} w_{n''n}^m \frac{d}{d\mu} P_{n''}^m(\mu), \tag{30}$$

on both sides

$$\frac{(1-\alpha\tilde{\mu})^2}{(1-\alpha^2)(1-\tilde{\mu}^2)} \left[\frac{(n+1)(n+m)}{2n+1} P_{n-1}^m(\mu') - \frac{n(n-m+1)}{2n+1} P_{n+1}^m(\mu')\right] =$$
$$\frac{1}{1-\mu^2}\left[\sum_{n''} w_{n''n}^m \frac{(n''+1)(n''+m)}{2n''+1} P_{n''-1}^m(\mu) - \sum_{n''m} w_{n''n}^m \frac{n''(n''-m+1)}{2n''+1} P_{n''+1}^m(\mu)\right]. \tag{31}$$

With $1-\mu^2 = \frac{(1-\alpha^2)(1-\tilde{\mu}^2)}{(1-\alpha\tilde{\mu})^2}$ on the right, many things cancel

$$\frac{(n+1)(n+m)}{2n+1} P_{n-1}^m(\tilde{\mu}) - \frac{n(n-m+1)}{2n+1} P_{n+1}^m(\tilde{\mu}) =$$
$$\sum_{n''} w_{n''n}^m \frac{(n''+1)(n''+m)}{(2n''+1)} P_{n''-1}^m(\mu) - \sum_{n''} w_{n''n}^m \frac{n''(n''-m+1)}{(2n''+1)} P_{n''+1}^m(\mu), \tag{32}$$

and inserting the unknown re-expansion Eq.(25) on the left hand side yields

$$\frac{(n+1)(n+m)}{2n+1} \sum_{n''} w_{n'',n-1}^m P_{n''}^m(\mu) - \frac{n(n-m+1)}{2n+1} \sum_{n''} w_{n'',n+1}^m P_{n''}^m(\mu) =$$
$$\sum_{n''} \frac{(n''+1)(n''+m)}{(2n''+1)} P_{n''-1}^m(\mu) w_{n''n}^m - \sum_{n''} \frac{n''(n''-m+1)}{(2n''+1)} P_{n''+1}^m(\mu) w_{n''n}^m. \tag{33}$$

Using the orthogonality relation by integration over $P_{n'}^m d\mu \int_{-1}^1$, each sum reduces to one term

$$\frac{(n+1)(n+m)}{2n+1} w_{n',n-1}^m - \frac{n(n-m+1)}{2n+1} w_{n',n+1}^m = \frac{(n'+2)(n'+m+1)}{(2n'+3)} w_{n'+1,n}^m - \frac{(n'-1)(n'-m)}{(2n'-1)} w_{n'-1,n}^m. \tag{34}$$

Recurrence 2. Multiplying Eq. (25) by $\tilde{\mu}$ yields with the recurrence Eq. (27) on the left hand side

$$\tilde{\mu} P_n^m(\tilde{\mu}) = \frac{n+m}{2n+1} P_{n-1}^m(\tilde{\mu}) + \frac{n-m+1}{2n+1} P_{n+1}^m(\tilde{\mu}), \tag{35}$$

$$(\mu+\alpha) P_n^m(\tilde{\mu}) = (1+\alpha\mu)\left(\frac{n+m}{2n+1} P_{n-1}^m(\tilde{\mu}) + \frac{n-m+1}{2n+1} P_{n+1}^m(\tilde{\mu})\right), \tag{36}$$

and leaves a factor on the un-warped side

$$(\mu+\alpha)\sum_{n''} w_{n''n}^m P_{n''}^m(\mu) = (1+\alpha\mu)\left(\frac{n+m}{2n+1}\sum_{n''} w_{n''n-1}^m P_{n''}^m(\mu) + \frac{n-m+1}{2n+1}\sum_{n''} w_{n''n+1}^m P_{n''}^m(\mu)\right). \tag{37}$$

$$\sum_{n''} w_{n''n}^m \left(\frac{n''+m}{2n''+1} P_{n''-1}^m + \alpha P_{n''}^m(\mu) + \frac{n''-m+1}{2n''+1} P_{n''+1}^m\right) =$$
$$\frac{n+m}{2n+1}\sum_{n''} w_{n''n-1}^m \left(\frac{\alpha(n''+m)}{2n''+1} P_{n''-1}^m(\mu) + P_{n''}^m(\mu) + \frac{\alpha(n''-m+1)}{2n''+1} P_{n''+1}^m(\mu)\right)$$
$$+ \frac{n-m+1}{2n+1}\sum_{n''} w_{n''n+1}^m \left(\frac{\alpha(n''+m)}{2n''+1} P_{n''-1}^m(\mu) + P_{n''}^m(\mu) + \frac{\alpha(n''-m+1)}{2n''+1} P_{n''+1}^m(\mu)\right), \tag{38}$$

Using the orthogonality relation by integration over $P_{n'}^m d\mu \int_{-1}^1$, each sum reduces

$$\begin{aligned}\tfrac{n'+m+1}{2n'+3} w^m_{n'+1\,n} + \alpha w^m_{n'n} + \tfrac{n'-m}{2n'-1} w^m_{n'-1\,n} = \\ \tfrac{n+m}{2n+1}\left(\tfrac{\alpha(n'+m+1)}{2n'+3} w^m_{n'+1\,n-1} + w^m_{n'\,n-1} + \tfrac{\alpha(n'-m)}{2n'-1} w^m_{n'-1\,n-1}\right) \\ + \tfrac{n-m+1}{2n+1}\left(\tfrac{\alpha(n'+m+1)}{2n'+3} w^m_{n'+1\,n+1} + w^m_{n'\,n+1} + \tfrac{\alpha(n'-m)}{2n'-1} w^m_{n'-1\,n+1}\right).\end{aligned} \tag{39}$$

Recurrence 3. The third recurrence Eq. (28) inserted into Eq. (25) yields

$$\begin{aligned}\tfrac{1}{\sqrt{1-\tilde{\mu}^2}}\left[\tfrac{(n-m+1)(n-m+2)}{(2n+1)} P^{m-1}_{n-1}(\tilde{\mu}) - \tfrac{(n+m-1)(n+m)}{(2n+1)} P^{m-1}_{n+1}(\tilde{\mu})\right] = \\ \tfrac{1}{\sqrt{1-\mu^2}} \textstyle\sum_{n''} w^m_{n''n}\left[\tfrac{(n''-m+1)(n''-m+2)}{2n''+1} P^{m+1}_{n''+1}(\mu) - \tfrac{(n''+m-1)(n''+m)}{2n''+1} P^{m+1}_{n''-1}(\mu)\right].\end{aligned} \tag{40}$$

With the property of the bilinear transform

$$\sqrt{\tfrac{1-\tilde{\mu}^2}{1-\mu^2}} = \sqrt{\tfrac{1-\frac{\mu^2+2\mu\alpha+\alpha^2}{1+2\mu\alpha+\mu^2\alpha^2}}{1-\mu^2}} = \sqrt{\tfrac{1+2\mu\alpha-2\mu\alpha-\mu^2-\alpha^2}{(1+\alpha\mu)^2(1-\mu^2)}} = \tfrac{\sqrt{1-\alpha^2}}{1+\alpha\mu}, \tag{41}$$

the divisor simplifies to a linear term

$$\begin{aligned}(1+\alpha\mu)\left[\tfrac{(n-m+1)(n-m+2)}{(2n+1)} P^{m-1}_{n-1}(\tilde{\mu}) - \tfrac{(n+m-1)(n+m)}{(2n+1)} P^{m-1}_{n+1}(\tilde{\mu})\right] = \\ \sqrt{1-\alpha^2} \textstyle\sum_{n''} w^m_{n''n}\left[\tfrac{(n''-m+1)(n''-m+2)}{2n''+1} P^{m+1}_{n''+1}(\mu) - \tfrac{(n''+m-1)(n''+m)}{2n''+1} P^{m+1}_{n''-1}(\mu)\right].\end{aligned} \tag{42}$$

Inserting the unknown re-expansion Eq. (25) yields

$$\begin{aligned}(1+\alpha\mu)\left[\tfrac{(n-m+1)(n-m+2)}{(2n+1)} \textstyle\sum_{n''} w^{m-1}_{n''n+1} P^{m-1}_{n''}(\mu) - \tfrac{(n+m-1)(n+m)}{(2n+1)} \textstyle\sum_{n''} w^{m-1}_{n''n-1} P^{m-1}_{n''}(\mu)\right] = \\ \sqrt{1-\alpha^2} \textstyle\sum_{n''} w^m_{n''n}\left[\tfrac{(n''-m+1)(n''-m+2)}{2n''+1} P^{m+1}_{n''+1}(\mu) - \tfrac{(n''+m-1)(n''+m)}{2n''+1} P^{m+1}_{n''-1}(\mu)\right].\end{aligned} \tag{43}$$

Using Eq. (27) replaces the linear multiplier μ

$$\begin{aligned}\tfrac{(n-m+1)(n-m+2)}{(2n+1)} \textstyle\sum_{n''} w^{m-1}_{n''n+1}\left[\tfrac{\alpha(n''+m-1)}{2n''+1} P^{m-1}_{n''-1}(\mu) + P^{m-1}_{n''}(\mu) + \tfrac{\alpha(n''-m+2)}{2n''+1} P^{m-1}_{n''+1}(\mu)\right] \\ - \tfrac{(n+m-1)(n+m)}{(2n+1)} \textstyle\sum_{n''} w^{m-1}_{n''n-1}\left[\tfrac{\alpha(n''+m-1)}{2n''+1} P^{m-1}_{n''-1}(\mu) + P^{m-1}_{n''}(\mu) + \tfrac{\alpha(n''-m+2)}{2n''+1} P^{m-1}_{n''+1}(\mu)\right] = \\ \sqrt{1-\alpha^2} \textstyle\sum_{n''} w^m_{n''n}\left[\tfrac{(n''-m+1)(n''-m+2)}{2n''+1} P^{m+1}_{n''+1}(\mu) - \tfrac{(n''+m-1)(n''+m)}{2n''+1} P^{m+1}_{n''-1}(\mu)\right].\end{aligned} \tag{44}$$

Using the orthogonality relation by integration over $P_{n'}^m d\mu \int_{-1}^1$, each sum reduces

$$\begin{aligned}\tfrac{(n-m+1)(n-m+2)}{(2n+1)}\left[\tfrac{\alpha(n'+m)}{2n'+3} w^{m-1}_{n'+\,n+1} + w^{m-1}_{n'n+1} + \tfrac{\alpha(n'-m+1)}{2n'-1} w^{m-1}_{n'-1\,n+1}\right] \\ - \tfrac{(n+m-1)(n+m)}{(2n+1)}\left[\tfrac{\alpha(n'+m)}{2n'+3} w^{m-1}_{n'+1\,n-1} + w^{m-1}_{n'\,n-1} + \tfrac{\alpha(n'-m+1)}{2n'-1} w^{m-1}_{n'-1\,n-1}\right] = \\ \sqrt{1-\alpha^2}\left[\tfrac{(n'-m)(n'-m+1)}{2n'-1} w^m_{n'-1\,n} - \tfrac{(n'+m)(n'+m+1)}{2n'+3} w^m_{n'+1\,n}\right].\end{aligned} \tag{45}$$

B ANGULAR LOUDNESS MODIFICATION

Utilizing the recurrence relation. Inserting the recurrence relation given in (Gumerov and Duraiswami, Zotter) in Eq. (22)

$$\mu\, P_n^m(\mu) = \frac{(n+m)\, P_{n-1}(\mu) + (n-m+1)\, P_{n+1}(\mu)}{2n+1},$$

the expression $g^m_{n'n} = \frac{(n'+m)!(2n'+1)}{2(n'-m)!} \int_{-1}^1 \frac{1+\alpha\mu}{\sqrt{1+\alpha^2}}\, P^m_n(\mu)\, P^m_{n'}(\mu)\, d\mu$ can be expanded to

$$g^m_{n'n} = \tfrac{(n'+m)!(2n'+1)}{2(n'-m)!} \int_{-1}^1 \left(\frac{\alpha(n+m)}{(2n+1)\sqrt{1+\alpha^2}}\, P^m_{n-1}(\mu) + P^m_n(\mu) + \frac{\alpha(n-m+1)}{(2n+1)\sqrt{1+\alpha^2}}\, P^m_{n+1}(\mu)\right) P^m_{n'}(\mu)\, d\mu. \tag{46}$$

Due to the orthogonality of the Legendre-functions $\frac{(n'-m)!(2n'+1)}{(n'+m)!}\int_{-1}^{1} P_{n'}^m P_n^m \, d\mu = \delta_{n'n}$:

$$\sqrt{1-\alpha^2}\; g_{n'n}^m = \frac{\alpha(n+m)}{2n+1}\delta_{n',n-1} + \delta_{n'n} + \frac{\alpha(n-m+1)}{2n+1}\delta_{n',n+1}. \tag{47}$$

For the conversion of given coefficients ϕ_n^m, this means

$$\tilde{\phi}_n^m = \frac{\alpha\,(n+m)}{(2n+1)\sqrt{1+\alpha^2}}\,\phi_{n-1}^m + \frac{1}{\sqrt{1+\alpha^2}}\phi_n^m + \alpha\,\frac{n-m+1}{(2n+1)\sqrt{1+\alpha^2}}\,\phi_{n+1}^m. \tag{48}$$

Inverse emphasis. Given an $\tilde{\mathrm{N}}^{\mathrm{th}}$ order, one can be assured of $\tilde{\phi}_{\tilde{\mathrm{N}}+1}^m = 0$ and $\tilde{\phi}_{\tilde{\mathrm{N}}+2}^m$ are always zero. Consequently, on could attempt to calculate a signal with inverse emphasis $\tilde{\phi}_n^m$ using

$$\tilde{\phi}_{\tilde{\mathrm{N}}}^m = \frac{\alpha\,(\tilde{\mathrm{N}}+m)}{(2\tilde{\mathrm{N}}+1)\sqrt{1+\alpha^2}}\,\phi_{\tilde{\mathrm{N}}-1}^m + \frac{1}{\sqrt{1+\alpha^2}}\phi_{\tilde{\mathrm{N}}}^m + \alpha\,\cancel{\frac{\tilde{\mathrm{N}}-m+1}{(2\tilde{\mathrm{N}}+1)\sqrt{1+\alpha^2}}\,\phi_{\tilde{\mathrm{N}}+1}^m}, \tag{49}$$

which, however, does not yield one, but two expressions. Inserting $\tilde{\mathrm{N}}+1$ reveals how one of these coefficients must behave:

$$\cancel{\tilde{\phi}_{\tilde{\mathrm{N}}+1}^m} = \frac{\alpha\,(\tilde{\mathrm{N}}+m+1)}{(2\tilde{\mathrm{N}}+3)\sqrt{1+\alpha^2}}\,\phi_{\tilde{\mathrm{N}}}^m + \frac{1}{\sqrt{1+\alpha^2}}\phi_{\tilde{\mathrm{N}}+1}^m + \alpha\,\cancel{\frac{\tilde{\mathrm{N}}-m+2}{(2\tilde{\mathrm{N}}+3)\sqrt{1+\alpha^2}}\,\phi_{\tilde{\mathrm{N}}+2}^m}, \tag{50}$$

but not usefully. In fact, it seem that for a general solution there must be an infinite number of non-zero coefficients for inverse emphasis.

De-emphasis. Nevertheless, if the present signal is known to result from emphasis, one could argue that before emphasis its order has been lower $\tilde{\mathrm{N}} = \mathrm{N}+1$. In this case, it is possible without errors to determine ϕ_n^m by knowing $\phi_n^m = 0$, $\forall n \geq \tilde{\mathrm{N}}$. Therefore,

$$\phi_{\tilde{\mathrm{N}}-1}^m = \frac{(2\tilde{\mathrm{N}}+1)\sqrt{1+\alpha^2}}{\alpha\,(\tilde{\mathrm{N}}+m)}\,\tilde{\phi}_{\tilde{\mathrm{N}}}^m, \tag{51}$$

$$\phi_{\tilde{\mathrm{N}}-2}^m = \frac{(2\tilde{\mathrm{N}}-1)\sqrt{1+\alpha^2}}{\alpha\,(\tilde{\mathrm{N}}+m-1)}\,\tilde{\phi}_{\tilde{\mathrm{N}}-1}^m - \frac{(2\tilde{\mathrm{N}}-1)}{\alpha\,(\tilde{\mathrm{N}}+m-1)}\,\phi_{\tilde{\mathrm{N}}-1}^m, \tag{52}$$

$$\phi_{n-1}^m = \frac{(2n+1)\sqrt{1+\alpha^2}}{\alpha\,(n+m)}\,\tilde{\phi}_n^m - \frac{(2n+1)}{\alpha\,(n+m)}\,\phi_n^m - \frac{n-m+1}{n+m}\,\phi_{n+1}^m, \qquad \text{for } n \leq \tilde{\mathrm{N}}-2. \tag{53}$$