ANALOGIES AND DIFFERENCES BETWEEN THREE METHODS FOR SOUND FIELD REPRODUCTION

Filippo M. Fazi[1], Philip A. Nelson[1] and Roland Potthast[2]

[1] Institute of Sound and Vibration Research, University of Southampton, U.K. (ff1@isvr.soton.ac.uk)

[2] Department of Mathematics, University of Reading, U.K.

Abstract: *A theoretical comparison is undertaken, between three approaches for the reproduction of a sound field. The first method is based on the decomposition of the sound field in terms of spherical harmonics (as High Order Ambisonics), the second method is derived from the Kirchhoff-Helmholtz integral (as Wave Field Synthesis) and the third approach relies on the solution of Least Squares algorithm. The study focuses on reproduction systems including a spherical array of secondary sources. The study is purely analytical, and is carried out in the framework of the theory of integral equations. Analogies and differences between the three techniques are discussed and the important link between the theory of sound field reconstruction and the theory of acoustic scattering is presented.*

1 INTRODUCTION

Many methods have been proposed, which attempt the reproduction of a desired sound field using an array of loudspeakers. Most of these methods are grounded on rigorous theoretical basis, which allow the computation of the driving signals of the loudspeakers from the knowledge of the target sound field.

One class of these methods includes the representation of the desired and reproduced sound field in terms of spherical harmonics. This category includes Ambisonics, which was initially proposed by Gerzon [1] and has been further developed into High Order Ambisonics [2], but also includes other techniques such as those presented, for example, in [3], [4] and [5].

The second class of techniques includes the formulation of the problem in terms of the Kirchhoff-Helmholtz integral or of the Rayleigh first integral, which explicitly define the loudspeaker signals in terms of the normal derivative of the desired sound field. The well established technique known as Wave Field Synthesis, which was first proposed by Berkhout [6], belongs to this category.

The third class of techniques is grounded on the numerical solution of an acoustical inverse problem, using techniques related to those used for the active control of sound [7], [8], [9], [10], [11], [12], [13] and [14].

In this paper the theoretical problem of sound field reproduction is tackled using a formulation based on a single layer potential [15] defined over a spherical surface. This approach is analogous to that described in [16]. We use the jump relation of the single layer potential to compute an exact and general solution of the problem under consideration. It is shown that this solution is closely related to an acoustical scattering problem. This solution will be used as a starting point for the theoretical analysis of some similarities and differences between the three families of techniques introduced above.

The loudspeaker array is mathematically represented as a continuous distribution of monopole-like sources (hereafter referred to as secondary sources),

arranged on a spherical surface in the free field. The spherical surface is represented by $\partial\Omega$, and the reproduction of the desired field is attempted in its interior region $\Omega \subset \mathbb{R}^3$.

It is important to mention that the ideal assumption defined above does not take into consideration the fact that, in all practical applications, the loudspeaker array includes a finite number of secondary sources. This fact is strictly related to the well-known problem known as spatial aliasing. The latter has been extensively studied in the literature and is of large relevance for a complete analysis of any sound field reproduction system. In fact, the spatial aliasing is the cause of artifacts, which can severely degrade the performance of the system when the wavelength of the sound to be reproduced is small in comparison to the average distance between the secondary sources. This paper is focused on the analogies and differences between the continuous formulations of the different reproduction techniques introduced above. For this reason, the issue of spatial aliasing is not addressed here and the reader is referred, for example, to [17], [18] and [19] for the analysis of this phenomenon in the framework of sound field reproduction.

As it has been mentioned above, the directivity of the secondary sources is not taken into consideration, as they are assumed to radiate sound as monopole sources. As a consequence of this and of the free field assumption, the sound field generated by each secondary source can be mathematically described by a free field Green function, defined by

$$G(\mathbf{x}, \mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \qquad (1)$$

Finally, the analysis is performed for a monochromatic sound field, with operating frequency ω. The desired field $p(\mathbf{x})$ is assumed to satisfy the homogeneous Helmholtz equation

$$\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x}) = 0 \qquad (2)$$

in the reproduction region Ω. In the two equations above, the wave number $k = \omega/c$, where c is the speed of sound, assumed to be uniform in $\mathbb{R}^3$.

Notation

Bold characters represent vectors in $\mathbb{R}^3$. We define

$$x := |\mathbf{x}|, \quad \hat{\mathbf{x}} := \frac{\mathbf{x}}{x} \qquad (3)$$

The relation between cartesian and polar coordinates of a vector $\mathbf{x}$ is given by

$$\begin{aligned} \mathbf{x} &= [x_1, \ x_2, \ x_3] \qquad (4) \\ &= [x\cos\phi_x \sin\theta_x, \ x\sin\phi_x \sin\theta_x, \ x\cos\theta_x] \end{aligned}$$

Given the open set Ω, its boundary is represented by $\partial\Omega$ and $\overline{\Omega} := \Omega \cup \partial\Omega$.

We also recall here the following orthogonality relation for the spherical harmonics [20]

$$\int_{\partial\Omega} Y_n^m(\hat{\mathbf{x}}) Y_{n'}^{m'}(\hat{\mathbf{x}}) dS(\mathbf{x}) \qquad (5)$$
$$= \int_0^{2\pi} d\phi_x \int_0^\pi Y_n^m(\hat{\mathbf{x}}) Y_{n'}^{m'}(\hat{\mathbf{x}}) r^2 sin(\theta_x) d\theta_x$$
$$= r^2 \delta_{nn'} \delta_{mm'}$$

and the following Wronskian relation [20]

$$j_n(x)h_n'(x) - j_n'(x)h_n(x) = \frac{i}{x^2} \qquad (6)$$

2 SINGLE LAYER POTENTIAL AND JUMP RELATION

The sound field generated by an infinite number of monopole-like secondary sources continuously arranged on $\partial\Omega$ can be mathematically represented by the following integral:

$$(Sa)(\mathbf{x}) := \int_{\partial\Omega} G(\mathbf{x}, \mathbf{y}) a(\mathbf{y}) dS(\mathbf{y}), \qquad (7)$$
$$\mathbf{x} \in \mathbb{R}^3 \backslash \partial\Omega$$

This integral is known as *single layer potential* [15]. The function $a(\mathbf{y})$ is usually referred to as the density of the potential and represents here the strength of the secondary sources. It is assumed that the integration is performed on the surface of a sphere with radius r. In this case, the infinitesimal portion of the surface $dS(\mathbf{y})$ can be expressed in spherical coordinates by $dS(\mathbf{y}) = r^2 \sin\theta d\phi_y d\theta_y$.

The single layer potential satisfies the homogeneous Helmholtz equation (1) both in the interior and exterior of regions Ω and $\mathbb{R}^3 \backslash \Omega$, respectively [21]. The integral (7) can be interpreted as the solution (non necessarily unique) of an *interior* Neumann problem in Ω and as the solution of an *exterior* Neumann problem in $\mathbb{R}^3 \backslash \overline{\Omega}$. For a definition of a Neumann boundary-value problem the reader can refer to [20] or [15]. It is therefore possible to define

$$p_i(\mathbf{x}) := (Sa)(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{8}$$

$$p_e(\mathbf{x}) := (Sa)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3 \backslash \overline{\Omega} \tag{9}$$

$p_i(\mathbf{x})$ and $p_e(\mathbf{x})$ are hereafter referred to as the interior field and exterior field, respectively. The interior field can be interpreted as the sound field due to sources located in the exterior of Ω, while the exterior field can be interpreted as a field due to sources in Ω.

If the density $a(\mathbf{y})$ is continuous, the single layer potential (7) is continuous throughout $\mathbb{R}^3$ [21], implying that

$$p_i(\mathbf{y}) = p_e(\mathbf{y}), \quad \mathbf{y} \in \partial\Omega \tag{10}$$

p_i and p_e represent two different fields, defined on different domains and with different physical nature. Even though the two fields have the same value on $\partial\Omega$, the values of their normal derivatives $\nabla_{\mathbf{n}} p_i$ and $\nabla_{\mathbf{n}} p_e$ are in general different. These are defined by

$$\nabla_{\mathbf{n}} p_i(\mathbf{y}) := \lim_{h \to 0^+} \hat{\mathbf{n}} \cdot \nabla p_i(\mathbf{y} - h\hat{\mathbf{n}}) \tag{11}$$

$$\nabla_{\mathbf{n}} p_e(\mathbf{y}) := \lim_{h \to 0^+} \hat{\mathbf{n}} \cdot \nabla p_e(\mathbf{y} + h\hat{\mathbf{n}}) \tag{12}$$

$$\mathbf{y} \in \partial\Omega$$

where $\hat{\mathbf{n}}$ is the unitary vector normal to $\partial\Omega$ and directed towards its exterior. The difference, or *jump*, between the normal derivatives is physically due to the presence of the layer of secondary sources on $\partial\Omega$, which determine a discontinuity in the gradient of the single layer potential (7).

The *jump relation* for the single layer potential represents a useful results, which relates the normal derivatives introduced above to the density $a(\mathbf{y})$ of the potential. This relation is given by [15]

$$a(\mathbf{y}) = \nabla_{\mathbf{n}} p_i(\mathbf{y}) - \nabla_{\mathbf{n}} p_e(\mathbf{y}) \tag{13}$$

If we choose the interior field $p_i(\mathbf{x})$ to be equal to the desired sound field $p(\mathbf{x})$ in Ω, then the jump relation provides the expression for the secondary source strength function $a(\mathbf{y})$, which allows a perfect reproduction of the desired field in Ω. It is therefore possible to write the following expression

$$p(\mathbf{x}) = \int_{\partial\Omega} G(\mathbf{x}, \mathbf{y})[\nabla_{\mathbf{n}} p_i(\mathbf{y}) - \nabla_{\mathbf{n}} p_e(\mathbf{y})]dS(\mathbf{y}),$$
$$\mathbf{x} \in \Omega \tag{14}$$

This result is perfectly consistent with the *Simple Source Formulation* presented in [20]. The arguments shown here represent a different method to obtain the same result. In order to compute the source strength from the complete knowledge of the desired sound field, we need to compute the normal derivative of the exterior field, $\nabla_{\mathbf{n}} p_e(\mathbf{y})$.

3 EQUIVALENT SCATTERING PROBLEM

The problem discussed in the previous section can be reformulated as an equivalent scattering problem. It has been observed that, assuming that $a(\mathbf{y})$ is continuous, the value of the exterior field p_e on $\partial\Omega$ equals the value of the interior field p_i. We assume now that $\partial\Omega$ does not represent anymore the loudspeaker array, but it represents instead the boundary of a scattering object. If the desired sound field $p(\mathbf{x})$ impinges on this scattering object, a *scattered sound field* $p_s(\mathbf{x})$ is generated. This field is a radiating solution of the Helmholtz equation [21], thus representing the solution to an exterior problem. The sum of the target (incident) field $p(\mathbf{x})$ and of the scattered field $p_s(\mathbf{x})$ gives the *total field* $p_T(\mathbf{x})$.

The scattering object Ω is assumed to be a *sound soft object*, or in other words an idealized object with pressure release boundaries. In mathematical terms, this corresponds to the condition that

$$p_T(\mathbf{y}) = 0, \quad \mathbf{y} \in \partial\Omega \tag{15}$$

that is the acoustic pressure of the total sound field equals zero on the boundary of the scattering object. This condition is referred to as the *homogeneous Dirichlet boundary condition*.

Under these assumptions and given a target/incident field $p(\mathbf{x})$, it can be easily seen that

$$p_s(\mathbf{y}) = -p(\mathbf{y}), \quad \mathbf{y} \in \partial\Omega \qquad (16)$$

Recalling the definition of the exterior field given in the previous section and in view of equation (10) and of the uniqueness of the exterior Dirichlet problem [15], it can be easily seen that

$$p_s(\mathbf{x}) = -p_e(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3 \backslash \Omega \qquad (17)$$

In view of this result and of the definition of $p_T(\mathbf{x})$ given above, it is possible to rewrite equation (14) as follows

$$\int_{\partial\Omega} G(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{n}} p_T(\mathbf{y}) dS(\mathbf{y}) \qquad (18)$$
$$= \begin{cases} p(\mathbf{x}), & \mathbf{x} \in \overline{\Omega} \\ -p_s(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^3 \backslash \Omega \end{cases}$$

This meaningful result can be summarized by the following sentence:

Given a desired field $p(\mathbf{x})$ and a continuous distribution of monopole-like sources on $\partial\Omega$, we want to compute the function $a(\mathbf{y})$, representing the strength of these sources, which allows for an exact reproduction of the desired field in Ω. This function is equal to the normal derivative of the total field $p_T(\mathbf{x})$ on $\partial\Omega$, which is generated by the scattering of the desired field by a sound soft object with the shape of Ω.

Equation (18) gives also the expression of the sound field generated by the layer of secondary sources in the exterior region $\mathbb{R}^3 \backslash \Omega$. This field is equal to $-p_s$, that is the phase-reversed scattered field.

4 SPHERICAL HARMONICS AND ANALOGY WITH HIGH ORDER AMBISONICS

As shown in [20], the interior and exterior sound field can be expressed by means of spherical harmonics and spherical Bessel functions:

$$p_i(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn}(\omega) j_n(kx) Y_n^m(\hat{\mathbf{x}}) \quad (19)$$

$$p_e(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{mn}(\omega) h_n(kx) Y_n^m(\hat{\mathbf{x}}) \quad (20)$$

where $Y_n^m(\hat{\mathbf{x}})$ are spherical harmonics, defined as in [20] and $j_n(\cdot)$ and $h_n(\cdot)$ are spherical Bessel functions and spherical Hankel functions of the first kind, respectively. For sake of completeness, it is important to specify that the time convention $e^{-i\omega t}$ has been adopted (if the sign in the argument of this exponential were positive, Hankel functions of the second kind should be used).

The two equations above can be used for the representation of the field generated by the single layer potential (18). Equation (19) can be used for the representation of the target sound field $p(\mathbf{x})$, corresponding to the field in the interior region Ω, while equation (20) can be used for the representation of the field in the exterior region $\mathbb{R}^3 \backslash \overline{\Omega}$. It is clear that, in general, $A_{mn} \neq C_{mn}$. The combination of equations (19) and (20) with equation (10), which represents the continuity of the single layer potential over the boundary $\partial\Omega$, leads to the following relation:

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn}(\omega) j_n(kr) Y_n^m(\hat{\mathbf{x}}) \qquad (21)$$
$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{mn}(\omega) h_n(kr) Y_n^m(\hat{\mathbf{x}})$$

It is recalled that $\partial\Omega$ is a sphere with radius r. In view of the orthogonality relation of the spherical harmonics (5), the previous equation leads to

$$C_{mn}(\omega) = -\frac{j_n(kr)}{h_n(kr)} A_{mn}(\omega) \qquad (22)$$

This is the boundary condition of a sound-soft sphere (sometimes referred to as the *Dirichlet sphere*). Equation (13) can be rewritten as

$$a(\mathbf{y}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} k A_{mn}(\omega) Y_n^m(\hat{\mathbf{x}}) \qquad (23)$$
$$\cdot \left(j_n'(kr) - \frac{j_n(kr)}{h_n(kr)} h_n'(kr) \right)$$

Applying the Wronskian relation (6) we obtain

$$a(\mathbf{y}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{A_{mn}(\omega)}{ikr^2 h_n(kr)} Y_n^m(\hat{\mathbf{x}}) \qquad (24)$$

Analogous results have been derived by Poletti [4], Wu and Abhayapala [22], Ahrens and Spors [23] and

by the authors [24]. Consistently with the arguments discussed in the previous section, it can be shown that this equation also represents the normal derivative $\nabla_{\mathbf{n}} p_T$ of the total sound field introduced above on $\partial\Omega$. For the derivation of the field scattered by a sound-soft sphere, see for example [20] or [16].

We assume now that the desired sound field corresponds to the field generated by a virtual monopole-like point source on $\mathbf{z} \in \partial\Omega$. In this case, the coefficients of the interior field are given by [20]

$$p(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [ikh_n(kr)Y_n^m(\hat{\mathbf{z}})^*]j_n(kr)Y_n^m(\hat{\mathbf{x}}),$$
$$\mathbf{x} \in \Omega \tag{25}$$

Substituting the expression in the square brackets above to the coefficients A_{mn} in equation (24) we obtain

$$a(\mathbf{y}) = \frac{1}{r^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\hat{\mathbf{z}})^* Y_n^m(\hat{\mathbf{y}}) \tag{26}$$
$$= \frac{1}{r^2} \delta(\cos\theta_y - \cos\theta_z)\delta(\phi_y - \phi_z)$$

The second equality is due to the completeness relation of the spherical harmonics (see [20] p.191). If this density is plugged into the expression (7) of the single layer potential, it can be easily seen that, for the well-known properties of the Dirac delta, the reproduced field equals the desired field generated by the virtual point source.

For sake of mathematical rigor, we should emphasize here that the assumption of the density $a(\mathbf{y})$ being continuous does not hold anymore for this particular target field. However the singularity at $\mathbf{z}$ does not represent a serious theoretical issue for the arguments presented here. In fact, this results holds rigorously if the series (25), representing the target field, is truncated to any finite order N. The truncation of the series corresponds to a spatially low-passed approximation of the desired sound field, and it is often applied when dealing with a finite number of secondary sources. As it is mentioned later, this truncation to the order N leads on one side to a reduction of the reproduction accuracy, but increases on the other side the robustness of the system. Applying mathematical passages analogous to

those shown by Rafaely [25] to the truncated series (26), we obtain the following expression for the density $a(\mathbf{y})$

$$a_N(\mathbf{y}) = \frac{1}{r^2} \sum_{n=0}^{N} \sum_{m=-n}^{n} Y_n^m(\hat{\mathbf{z}})^* Y_n^m(\hat{\mathbf{y}}) \tag{27}$$
$$= \frac{1}{r^2} \sum_{n=0}^{N} \frac{2n+1}{4\pi} P_n(\hat{\mathbf{y}} \cdot \hat{\mathbf{z}})$$
$$= \frac{1}{r^2} \frac{N+1}{4\pi} \frac{P_N(\hat{\mathbf{y}} \cdot \hat{\mathbf{z}}) - P_{N+1}(\hat{\mathbf{y}} \cdot \hat{\mathbf{z}})}{1 - \hat{\mathbf{y}} \cdot \hat{\mathbf{z}}}$$

where the functions P_n are Legendre polynomials [20]. An identical result can be obtained if the ideal assumption is made that both the target field and the field generated by each secondary source can be approximated by a plane wave, with expansion coefficients $A_{mn} = 4\pi i^n Y_n^m(\hat{\mathbf{y}})^*$, where $\hat{\mathbf{y}}$ is the direction of propagation of the plane wave considered [20].

The first equality of equation (27) corresponds to a High Order Ambisonics panning, as reported for example by Daniel at al. [26], equation (16). With analogous passages, the direct relation can be derived between equation (24) and the Near-Field Compensated High Order Ambisonics decoding process [26].

5 HIGH FREQUENCY SCATTERING AND ANALOGY WITH WAVE FIELD SYNTHESIS

We consider now the case of an acoustic wave scattered by a pressure release infinite plane $\partial\Lambda$. We assume that the incident field is due to a monopole source with strength a_0 at a distance d, say, from the plane. It can be seen that the scattered sound field is equivalent to that generated by a mirror source, specular in respect to the scattering plane to the monopole source generating the incident field, but with phase-reversed strength $-a_0$. This scattered field satisfies the pressure release boundary condition $p_T = 0$ on the scattering plane. It follows that the normal derivative of the scattered acoustic field equals the normal derivative of the incident field. Therefore, the normal derivative of the total pressure field, $\nabla_{\mathbf{n}} p_T$, equals twice the normal derivative of

the incident field. This argument can be extended to a more general category of sound fields. If this result is plugged into the jump relation (13), we see that the single layer potential reduces to the first Rayleigh integral formula:

$$p(\mathbf{x}) = \int_{\partial\Lambda} G(\mathbf{x}, \mathbf{y}) 2\nabla_{\mathbf{n}} p(\mathbf{y}) dS(\mathbf{y}) \qquad (28)$$

It should be noticed that the integration is performed over the infinite plane $\partial\Lambda$ instead of over the sphere $\partial\Omega$ (in which case the above result would not hold).

Going back to the case of spherical geometry addressed in this paper, the result derived above proves to be useful when the wave length considered, $\lambda = 2\pi/k$, is much smaller than the radius r of the sphere $\partial\Omega$. In this case, as suggested by Colton and Kress [21] p.54, it is possible to solve the scattering problem using the *Kirchhoff approximation* of the total field. As a first step, it is assumed that p_T and its normal derivative $\nabla_{\mathbf{n}} p_T$ equal zero in the so-called *shadow region* $\partial\Omega_+$, that is the region which is not illuminated by the incident field. For example, in the case of an incident plane wave coming from the direction $[\theta_0 = \pi/2, \; \phi_0 = 0]$, the shadow region corresponds to the hemisphere $\partial\Omega_+ = \{\mathbf{y} \in \partial\Omega : \pi/2 < \phi_y < 3\pi/2\}$, while the illuminated area is $\partial\Omega_- = \{\mathbf{y} \in \partial\Omega : -\pi/2 \le \phi_y \le \pi/2\}$. In the illuminated area $\partial\Omega_-$, the scattering object can be considered locally as a plane. Therefore, as shown above, $\nabla_{\mathbf{n}} p_T(\mathbf{y}) = 2\nabla_{\mathbf{n}} p(\mathbf{y}), \; \mathbf{y} \in \partial\Omega_-$.

Inserting these results in equation 18, we obtain the following result:

$$p(\mathbf{x}) \approx \int_{\partial\Omega_-} G(\mathbf{x}, \mathbf{y}) 2\nabla_{\mathbf{n}} p(\mathbf{y}) dS(\mathbf{y}), \quad (29)$$
$$\mathbf{x} \in \Omega$$

This result is analogous to the Wave Field Synthesis approach with an analytical secondary source selection criterion, proposed by Spors [27], derived from the Kirchhoff-Helmholtz integral.

As mentioned above, this result is a high frequency approximation of the more general result given by equation (18), and does not produce an accurate reproduction of the desired field when the wave length considered is comparable with the size of Ω. On the

other side, this approach allows an explicit and simple computation of the source strength function $a(\mathbf{y})$ in terms of the normal derivative of the incident field, with no need to solve a (usually non trivial) scattering problem.

6 ACOUSTICAL INVERSE PROBLEM AND SOLUTION OF THE INTEGRAL EQUATION

It is now assumed that the desired sound field is not known *a priori*, but it is measured by an array of L omnidirectional ideal microphones, arranged at the locations $\mathbf{x}_l \in \overline{\Omega}$. We now want to define the source strength function $a(\mathbf{y})$ that minimizes the cost function J given by

$$J := \sum_{l=1}^{L} \left| p(\mathbf{x}_l) - \int_{\partial\Omega} G(\mathbf{x}_l, \mathbf{y}) a(\mathbf{y}) dS(\mathbf{y}) \right|^2$$
$$(30)$$

This error minimization problem is a generalization of the approach proposed by Kirkeby and Nelson [8], which is grounded on the solution of an inverse problem and is strictly related to the least-squares method often used for active control of sound [7].

The ideal assumption is made that an infinite number of microphones is arranged on the boundary $\partial\Omega$. As discussed in [28] and [16] if the desired sound field is reproduced exactly on the boundary $\partial\Omega$ and if the wave number k is not one of the so-called *Dirichlet eigenvalues* k_n for Ω, then the target field is reproduced accurately also in Ω. This result is due to the uniqueness and solvability of the interior Dirichlet problem for the case under consideration [15]. It should be considered that this one-to-one relation between the pressure field on the boundary and on the interior region does not hold if k is one of the Dirichlet eigenvalues mentioned above. These values correspond to the resonance frequencies of the sound soft sphere Ω and are identified by the zeros of the spherical Bessel functions, that is $j_n(k_n r) = 0$. As it will become more clear later, this is a non-uniqueness problem and can be overcome in different ways (not presented here).

What has been discussed above means, in other terms, that the control effort can be limited to the boundary $\partial\Omega$, and the control of the acoustic pres-

sure only (and not of its normal derivative) is sufficient (again, if $k \neq k_n$). Under the ideal assumption of a continuous distribution of microphones on $\partial\Omega$, equation (30) can be rewritten as

$$J = \int_{\partial\Omega} \left| p(\mathbf{x}) - \int_{\partial\Omega} G(\mathbf{x}, \mathbf{y})a(\mathbf{y})dS(\mathbf{y}) \right|^2 \frac{dS(\mathbf{x})}{4\pi r^2}$$
(31)

As it has been shown, under the assumption considered here the target field $p(\mathbf{x})$ is in the range of the single layer potential (7), implying that the minimum achievable error J equals zero. The secondary source strength function $a(\mathbf{y})$ is given by the jump relation (13). However, neither the normal derivative of the desired field nor the normal derivative of the corresponding exterior field are known now. This fact represents a complication in the determination of $a(\mathbf{y})$. The problem addressed can be reformulated in terms of the following integral equation

$$p(\mathbf{x}) = \int_{\partial\Omega} G(\mathbf{x}, \mathbf{y})a(\mathbf{y})dS(\mathbf{y}), \qquad (32)$$
$$\mathbf{x} \in \partial\Omega$$

where the function $p(\mathbf{x})$ is given on $\partial\Omega$ and $a(\mathbf{y})$ must be computed. The equation above represents an integral equation of the first kind, which is in general not solvable. We know that a solution $a(\mathbf{y})$ exists and is given by the jump relation (13). However, the determination of $a(\mathbf{y})$ is an ill-conditioned inverse problem. As shown by the authors [28], [16], it is possible to seek a solution to (32) in terms of the singular system of the integral operator involved. We give here the solution of this problem for the spherical geometry, and reference [29] for detail on the derivation:

$$a(\mathbf{y}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{Y_n^m(\hat{\mathbf{y}})}{ikr^4 j_n(kr)h_n(kr)} \qquad (33)$$
$$\cdot \int_{\partial\Omega} p(\mathbf{x})Y_n^m(\hat{\mathbf{x}})^* dS(\mathbf{x})$$

It can be observed that the ill-conditioned nature of the problem is given by the decay of the singular values of the integral operator, appearing in the denominator of the fraction in the equation above. However, the large order approximation of the spherical Bessel and Hankel functions (see, for example, [21] or [20])

shows that the singular values exhibit a linear decay (instead of exponential), thus implying that the inverse problem considered is only *mildly* ill-posed. It is possible to apply a regularization scheme in order to increase the robustness of the system against noise and other errors. Besides the usual Tikhonov regularization method, it is also possible to apply a simple spectral cut-off of the operator. The latter corresponds to the truncation of the series (35) to a given order N (as it was done for equation (27)).

It can be noticed that equation (24) and equation (35) are analogous, and their comparison leads to the following relation

$$A_{mn}(\omega) = \frac{1}{r^2 j_n(kr)} \int_{\partial\Omega} p(\mathbf{x})Y_n^m(\hat{\mathbf{x}})^* dS(\mathbf{x})$$
(34)

This result was also derived by Williams [20] and by Poletti [4]. Probably not surprisingly, this equation is analogous to the Ambisonics encoding equation for the so called open-sphere case (see Daniel et al. [26] equation (26) using omnidirectional microphones).

It can be seen that the problem with the Dirichlet eigenvalues is directly related to the fact that, for the presence of the spherical Bessel function, the denominator in the equation above can equal zero. As mentioned before, this occurs at some specific frequencies, sometimes referred to as *forbidden frequencies* [20]. This fact does not imply that the denominator of (34) equals zero for a given n, but it rather implies that the corresponding coefficients A_{mn} can not be determined uniquely. In fact, at the given Dirichlet eigenvalue k_ν any solution of the form

$$a(\mathbf{y}) = \sum_{\substack{n=0 \\ n\neq\nu}}^{\infty} \sum_{m=-n}^{n} \frac{Y_n^m(\hat{\mathbf{y}})}{ik_\nu r^4 j_n(k_\nu r)h_n(k_\nu r)} \quad (35)$$
$$\cdot \int_{\partial\Omega} p(\mathbf{x})Y_n^m(\hat{\mathbf{x}})^* dS(\mathbf{x})$$
$$+ \sum_{m=-\nu}^{\nu} b_m Y_\nu^m(\hat{\mathbf{y}}), \;\; b_m \in \mathbb{C}$$

is a solution to the integral equation (32). However, only one of these solutions allows an exact reproduction of the desired field in Ω.

7 CONCLUSION

The problem of the reproduction of a desired sound field with a continuous distribution of monopole-like secondary sources on a sphere has been addressed, and the function $a(\mathbf{y})$, representing the strength of these sources, has been calculated using the jump relation of the single layer potential. It has been shown that the solution of the sound field reconstruction problem is strictly related to the solution of an equivalent scattering problem involving a sound soft sphere. In particular, $a(\mathbf{y})$ has been shown to be equal to the normal derivative of the total field (incident field + scattered field).

It has been shown that the solution of this scattering problem for the spherical geometry considered can be expressed in terms of a series of spherical harmonics and spherical Bessel functions. This analytical expression of the secondary source strength function has been shown to be strictly related to the High Order Ambisonics decoding equation.

For wave length much smaller than the radius of the sphere $\partial\Omega$, the scattering problem can be solved using the Kirchhoff approximation. This implies that the secondary source strength function $a(\mathbf{y})$ is twice the normal derivative of the target field in the illuminated area and zero in the shadow area. The relation of this high frequency approximation method with Wave Field Synthesis has been highlighted.

Finally, starting from a least squared error approach, the sound field reproduction problem has been formulated as an integral equation of the first kind. Its solution, expressed in terms of the singular system of the integral operator involved, corresponds as expected to the function $a(\mathbf{y})$ computed with the jump relation of the single layer potential. The direct relation between the acoustical inverse problem addressed and the equivalent scattering problem has been emphasized.

Many of the results shown can be extended to other geometries of Ω, different from a sphere (as long as Ω is a bounded region with smooth boundaries). However, all the equations involving spherical harmonics and spherical Bessel functions are valid only for the spherical geometry considered here.

REFERENCES

[1] M. A. Gerzon, "Periphony - with-height sound reproduction," *Journal of the Audio Engineering Society*, vol. 21, no. 1, pp. 2–10, 1973.

[2] J. Daniel, *Représentation de Champs Acoustiques, Application à la Transmission et à la Reproduction de Scènes Sonores Complexes dans un Contexte Multimédia*. Phd thesis, 2000.

[3] D. B. Ward and T. D. Abhayapala, "Reproduction of a plane-wave sound field using an array of loudspeakers," *Ieee Transactions on Speech and Audio Processing*, vol. 9, no. 6, pp. 697–707, 2001.

[4] M. A. Poletti, "Three-dimensional surround sound systems based on spherical harmonics," *Journal of the Audio Engineering Society*, vol. 53, no. 11, pp. 1004–1025, 2005.

[5] J. Hannemann and K. D. Donohue, "Virtual sound source rendering using a multipole-expansion and method-of-moments approach," *Journal of the Audio Engineering Society*, vol. 56, no. 6, pp. 473–481, 2008.

[6] A. J. Berkhout, "A holographic approach to acoustic control," *Journal of the Audio Engineering Society*, vol. 36, no. 12, pp. 977–995, 1988.

[7] P. A. Nelson and S. J. Elliott, *Active control of sound*. London: Academic Press, 1992.

[8] O. Kirkeby and P. A. Nelson, "Reproduction of plane-wave sound fields," *Journal of the Acoustical Society of America*, vol. 94, no. 5, pp. 2992–3000, 1993.

[9] T. Betlehem and T. D. Abhayapala, "Theory and design of sound field reproduction in reverberant rooms," *Journal of the Acoustical Society of America*, vol. 117, no. 4, pp. 2100–2111, 2005.

[10] J. W. Choi and Y. H. Kim, "Generation of an acoustically bright zone with an illuminated region using multiple sources," *Journal of the Acoustical Society of America*, vol. 111, no. 4, pp. 1695–1700, 2002.

[11] N. Epain and E. Friot, "Active control of sound inside a sphere via control of the acoustic pressure at the boundary surface," *Journal of Sound and Vibration*, vol. 299, no. 3, pp. 587–604,

2007.

[12] S. Ise, "A principle of sound field control based on the kirchhoffâĂŞhelmholtz integral equation and the theory on inverse systems," *Acustica-Acta Acustica*, vol. 85, pp. 78–87, 1999.

[13] W. H. Cho, J. G. Ih, and M. M. Boone, "Holographic design of a source array for achieving a desired sound field," in *124th International Convention of the Audio Engineering Society*, (Amsterdam), 2008.

[14] S. Takane, Y. Suzuki, and T. Sone, "A new method for global sound field reproduction based on kirchhoff's integral equation," *Acustica*, vol. 85, no. 2, pp. 250–257, 1999.

[15] D. L. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*. New York: Wiley, 1983.

[16] F. M. Fazi, P. A. Nelson, J. E. N. Christensen, and J. Seo, "Surround system based on three-dimensional sound field reconstruction," in *125th International Convention of the Audio Engineering Society*, (San Francisco, USA), 2008.

[17] E. W. Start, *Direct Sound Enhancement by Wave Field Synthesis.* Phd thesis, 1997.

[18] E. Corteel, R. Pellegrini, and C. Kuhn-Rahloff, "Wave field synthesis with increased aliasing frequency," in *124th International Convention of the Audio Engineering Society*, (Amsterdam), 2008.

[19] S. Spors and R. Rabenstein, "Spatial aliasing artifacts produced by linear and circular loudspeaker arrays used for wave field synthesis," in *120th International Convention of the Audio Engineering Society*, (Paris, France), 2006.

[20] E. G. Williams, *Fourier acoustics : sound radiation and nearfield acoustical holography*. San Diego: Academic, 1999.

[21] D. L. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*. Applied mathematical sciences, Berlin: Springer, 1992.

[22] Y. J. Wu and T. D. Abhayapala, "Theory and design of soundfield reproduction using continuous loudspeaker concept," *Ieee Transactions on Audio Speech and Language Processing*, vol. 17, no. 1, pp. 107–116, 2009.

[23] J. Ahrens and S. Spors, "An analytical approach to sound field reproduction using circular and spherical loudspeaker distributions," *Acta Acustica United with Acustica*, vol. 94, no. 6, pp. 988–999, 2008.

[24] F. M. Fazi and P. A. Nelson, "A theoretical study of sound field reconstruction techniques," in *19th International Congress on Acoustics*, (Madrid), 2007.

[25] B. Rafaely, "Plane-wave decomposition of the sound field on a sphere by spherical convolution," *Journal of the Acoustical Society of America*, vol. 116, no. 4, pp. 2149–2157, 2004.

[26] J. Daniel, R. Nicol, and S. Moureau, "Further investigations of high-order ambisonics and wavefield synthesis for holophonic sound imaging," in *114th International Convention of the Audio Engineering Society*, 2003.

[27] S. Spors, "Extension of analytic secondary source selection criterion for wave field synthesis," in *123rd International Convention of the Audio Engineering Society*, (New York), 2007.

[28] F. M. Fazi and P. A. Nelson, "Application of functional analysis to the sound field reconstruction," in *23rd Conference on Reproduced Sound of the Institute of Acoustics*, (Newcastle), 2007.

[29] F. M. Fazi, P. A. Nelson, R. Potthast, and J. Seo, "An introduction to a generalised theory for sound field reproduction," in *24th Conference on Reproduced Sound of the Institute of Acoustics*, (Brighton), 2008.