Appendix A

Recursions of associated Legendre functions P_n^m :

$$P_0^0(x) = 1 \quad (1)$$

$$P_n^n(x) = (2n-1)\sqrt{1-x^2}\,P_{n-1}^{n-1}(x) \ \ldots \ \forall \ n \geq 1 \quad (2)$$

$$P_n^{n-1}(x) = (2n-1)xP_{n-1}^{n-1}(x) \ \ldots \ \forall \ n \geq 1 \quad (3)$$

$$P_n^m(x) = \tfrac{1}{n-m}\Big((2n-1)xP_{n-1}^m(x) - (n+m-1)P_{n-2}^m(x)\Big) \ \ldots \ \forall \ m \geq 0 \ \wedge \ n-2 \geq m \quad (4)$$

$n = 0,1,2,3,\ldots \qquad m = 0,\ldots,+n$

$x = \sin\vartheta$

ϑ ... elevation angle;

$P_0^0(\sin\vartheta) = 1$

$P_1^0(\sin\vartheta) = \sin\vartheta$

$P_1^1(\sin\vartheta) = -\cos\vartheta$

$P_2^0(\sin\vartheta) = \tfrac{1}{2}(3\cos^2\vartheta - 1)$

$P_2^1(\sin\vartheta) = -3\cos\vartheta\sin\vartheta$

$P_2^2(\sin\vartheta) = 3\cos^2\vartheta$

$P_3^0(\sin\vartheta) = \tfrac{1}{2}\sin\vartheta(5\sin^2\vartheta - 3)$

$P_3^1(\sin\vartheta) = -\tfrac{3}{2}\cos\vartheta(5\sin^2\vartheta - 1)$

$P_3^2(\sin\vartheta) = 15\cos^2\vartheta\sin\vartheta$

Condon-Shortley transformed associated Legendre functions $\tilde{P}_n^m$:

$$\tilde{P}_n^{|m|}(\sin\vartheta) = (-1)^m P_n^{|m|}(\sin\vartheta) \quad (5)$$

$n = 0,1,2,3,\ldots \qquad m = 0,\ldots,+n$

$\tilde{P}_0^0(\sin\vartheta) = 1$

$\tilde{P}_1^0(\sin\vartheta) = \sin\vartheta$

$\tilde{P}_1^1(\sin\vartheta) = \cos\vartheta$

$\tilde{P}_2^0(\sin\vartheta) = \tfrac{1}{2}(3\cos^2\vartheta - 1)$

$\tilde{P}_2^1(\sin\vartheta) = 3\cos\vartheta\sin\vartheta$

$\tilde{P}_2^2(\sin\vartheta) = 3\cos^2\vartheta$

$\tilde{P}_3^0(\sin\vartheta) = \tfrac{1}{2}\sin\vartheta(5\sin^2\vartheta - 3)$

$\tilde{P}_3^1(\sin\vartheta) = \tfrac{3}{2}\cos\vartheta(5\sin^2\vartheta - 1)$

$\tilde{P}_3^2(\sin\vartheta) = 15\cos^2\vartheta\sin\vartheta$

$\tilde{P}_3^3(\sin\vartheta) = 15\cos^3\vartheta$

orthonormalized complex spherical harmonics Y_n^m:

$$Y_n^m(\vartheta,\varphi)=\sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\tilde{P}_n^{|m|}(\sin\vartheta)e^{im\varphi} \tag{6}$$

$n=0,1,2,3,\ldots \qquad m=-n,\ldots,+n$

φ ... azimuth angle

orthonormalized real spherical harmonics $\hat{Y}_n^m$:

$$\hat{Y}_n^{+m}(\vartheta,\varphi)=\sqrt{\frac{(2-\delta_m)(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}\tilde{P}_n^m(\sin\vartheta)\cos(m\varphi) \tag{7}$$

$n=0,1,2,3,\ldots \qquad m=0,\ldots,+n$

$$\hat{Y}_n^{-m}(\vartheta,\varphi)=\sqrt{\frac{(2-\delta_m)(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}\tilde{P}_n^m(\sin\vartheta)\sin(m\varphi) \tag{8}$$

$n=1,2,3,\ldots \qquad m=1,\ldots,+n$

δ_m ... Kronecker delta

$\hat{Y}_0^0(\sin\vartheta)=\sqrt{\tfrac{1}{4\pi}}$

$\hat{Y}_1^{-1}(\sin\vartheta)=\sqrt{\tfrac{3}{4\pi}}\cos\vartheta\sin\varphi$

$\hat{Y}_1^0(\sin\vartheta)=\sqrt{\tfrac{3}{4\pi}}\sin\vartheta$

$\hat{Y}_1^1(\sin\vartheta)=\sqrt{\tfrac{3}{4\pi}}\cos\vartheta\cos\varphi$

$\hat{Y}_2^{-2}(\sin\vartheta)=\sqrt{\tfrac{15}{16\pi}}\cos^2\vartheta\sin 2\varphi$

$\hat{Y}_2^{-1}(\sin\vartheta)=\sqrt{\tfrac{15}{4\pi}}\cos\vartheta\sin\vartheta\sin\varphi$

$\hat{Y}_2^0(\sin\vartheta)=\sqrt{\tfrac{5}{16\pi}}(3\cos^2\vartheta-1)$

$\hat{Y}_2^{+1}(\sin\vartheta)=\sqrt{\tfrac{15}{4\pi}}\cos\vartheta\sin\vartheta\cos\varphi$

$\hat{Y}_2^{+2}(\sin\vartheta)=\sqrt{\tfrac{15}{16\pi}}\cos^2\vartheta\cos 2\varphi$

$\hat{Y}_3^{-3}(\sin\vartheta)=\sqrt{\tfrac{35}{32\pi}}\cos^3\vartheta\sin 3\varphi$

$\hat{Y}_3^{-2}(\sin\vartheta)=\sqrt{\tfrac{105}{16\pi}}\cos^2\vartheta\sin\vartheta\sin 2\varphi$

$\hat{Y}_3^{-1}(\sin\vartheta)=\sqrt{\tfrac{21}{32\pi}}\cos\vartheta(5\sin^2\vartheta-1)\sin\varphi$

$\hat{Y}_3^0(\sin\vartheta)=\sqrt{\tfrac{7}{16\pi}}\sin\vartheta(5\sin^2\vartheta-3)$

$\hat{Y}_3^{+1}(\sin\vartheta)=\sqrt{\tfrac{21}{32\pi}}\cos\vartheta(5\sin^2\vartheta-1)\cos\varphi$

$\hat{Y}_3^{+2}(\sin\vartheta)=\sqrt{\tfrac{105}{16\pi}}\cos^2\vartheta\sin\vartheta\cos 2\varphi$

$\hat{Y}_3^{+3}(\sin\vartheta)=\sqrt{\tfrac{35}{32\pi}}\cos^3\vartheta\cos 3\varphi$

semi-normalized real spherical harmonics $\bar{Y}_n^m$:

$$\bar{Y}_n^m(\vartheta,\varphi) = \sqrt{\frac{4\pi}{2n+1}}\hat{Y}_n^m(\vartheta,\varphi) \qquad (9)$$

$n = 0,1,2,3,\ldots \qquad m = -n,\ldots,+n$

$\bar{Y}_0^0(\sin\vartheta) = 1$
$\bar{Y}_1^{-1}(\sin\vartheta) = \cos\vartheta\sin\varphi$
$\bar{Y}_1^0(\sin\vartheta) = \sin\vartheta$
$\bar{Y}_1^1(\sin\vartheta) = \cos\vartheta\cos\varphi$
$\bar{Y}_2^{-2}(\sin\vartheta) = \sqrt{\tfrac{3}{4}}\cos^2\vartheta\sin 2\varphi$
$\bar{Y}_2^{-1}(\sin\vartheta) = \sqrt{3}\cos\vartheta\sin\vartheta\sin\varphi$
$\bar{Y}_2^0(\sin\vartheta) = \tfrac{1}{2}(3\cos^2\vartheta - 1)$
$\bar{Y}_2^{+1}(\sin\vartheta) = \sqrt{3}\cos\vartheta\sin\vartheta\cos\varphi$
$\bar{Y}_2^{+2}(\sin\vartheta) = \sqrt{\tfrac{3}{4}}\cos^2\vartheta\cos 2\varphi$
$\bar{Y}_3^{-3}(\sin\vartheta) = \sqrt{\tfrac{5}{8}}\cos^3\vartheta\sin 3\varphi$
$\bar{Y}_3^{-2}(\sin\vartheta) = \sqrt{\tfrac{15}{4}}\cos^2\vartheta\sin\vartheta\sin 2\varphi$
$\bar{Y}_3^{-1}(\sin\vartheta) = \sqrt{\tfrac{3}{8}}\cos\vartheta(5\sin^2\vartheta - 1)\sin\varphi$
$\bar{Y}_3^0(\sin\vartheta) = \tfrac{1}{2}\sin\vartheta(5\sin^2\vartheta - 3)$
$\bar{Y}_3^{+1}(\sin\vartheta) = \sqrt{\tfrac{3}{8}}\cos\vartheta(5\sin^2\vartheta - 1)\cos\varphi$
$\bar{Y}_3^{+2}(\sin\vartheta) = \sqrt{\tfrac{15}{4}}\cos^2\vartheta\sin\vartheta\cos 2\varphi$
$\bar{Y}_3^{+3}(\sin\vartheta) = \sqrt{\tfrac{5}{8}}\cos^3\vartheta\cos 3\varphi$

not-normalized real spherical harmonics = reference spherical harmonics $\breve{Y}_n^m$:

$$\breve{Y}_n^m(\vartheta,\varphi) = \sqrt{\frac{4\pi}{2n+1}\frac{(n+|m|)!}{(n-|m|)!}}\hat{Y}_n^m(\vartheta,\varphi) \tag{10}$$

$$n = 0,1,2,3,\ldots \qquad m = -n,\ldots,+n$$

$$\breve{Y}_0^0(\sin\vartheta) = 1$$
$$\breve{Y}_1^{-1}(\sin\vartheta) = \cos\vartheta\sin\varphi$$
$$\breve{Y}_1^0(\sin\vartheta) = \sin\vartheta$$
$$\breve{Y}_1^1(\sin\vartheta) = \cos\vartheta\cos\varphi$$
$$\breve{Y}_2^{-2}(\sin\vartheta) = 3\cos^2\vartheta\sin 2\varphi$$
$$\breve{Y}_2^{-1}(\sin\vartheta) = 3\cos\vartheta\sin\vartheta\sin\varphi$$
$$\breve{Y}_2^0(\sin\vartheta) = \tfrac{1}{2}(3\cos^2\vartheta - 1)$$
$$\breve{Y}_2^{+1}(\sin\vartheta) = 3\cos\vartheta\sin\vartheta\cos\varphi$$
$$\breve{Y}_2^{+2}(\sin\vartheta) = 3\cos^2\vartheta\cos 2\varphi$$
$$\breve{Y}_3^{-3}(\sin\vartheta) = 15\cos^3\vartheta\sin 3\varphi$$
$$\breve{Y}_3^{-2}(\sin\vartheta) = 15\cos^2\vartheta\sin\vartheta\sin 2\varphi$$
$$\breve{Y}_3^{-1}(\sin\vartheta) = \tfrac{3}{2}\cos\vartheta(5\sin^2\vartheta - 1)\sin\varphi$$
$$\breve{Y}_3^0(\sin\vartheta) = \tfrac{1}{2}\sin\vartheta(5\sin^2\vartheta - 3)$$
$$\breve{Y}_3^{+1}(\sin\vartheta) = \tfrac{3}{2}\cos\vartheta(5\sin^2\vartheta - 1)\cos\varphi$$
$$\breve{Y}_3^{+2}(\sin\vartheta) = 15\cos^2\vartheta\sin\vartheta\cos 2\varphi$$
$$\breve{Y}_3^{+3}(\sin\vartheta) = 15\cos^3\vartheta\cos 3\varphi$$

orthonormalized weights ON_n^m related to reference spherical harmonics:

$$ON_0^0 = \sqrt{\tfrac{1}{4\pi}}; \tag{11}$$
$$ON_1^{-1} = \sqrt{\tfrac{3}{4\pi}},\ ON_1^0 = \sqrt{\tfrac{3}{4\pi}},\ ON_1^{+1} = \sqrt{\tfrac{3}{4\pi}};$$
$$ON_2^{-2} = \sqrt{\tfrac{5}{48\pi}},\ ON_2^{-1} = \sqrt{\tfrac{5}{12\pi}},\ ON_2^0 = \sqrt{\tfrac{5}{4\pi}},\ ON_2^{+1} = \sqrt{\tfrac{5}{12\pi}},\ ON_2^{+2} = \sqrt{\tfrac{5}{48\pi}};$$
$$ON_3^{-3} = \sqrt{\tfrac{7}{1440\pi}},\ ON_3^{-2} = \sqrt{\tfrac{7}{240\pi}},\ ON_3^{-1} = \sqrt{\tfrac{7}{24\pi}},\ ON_3^0 = \sqrt{\tfrac{7}{4\pi}},$$
$$ON_3^{+1} = \sqrt{\tfrac{7}{24\pi}},\ ON_3^{+2} = \sqrt{\tfrac{7}{240\pi}},\ ON_3^{+3} = \sqrt{\tfrac{7}{1440\pi}};$$

semi-normalized weights SN_n^m related to reference spherical harmonics:

$$SN_0^0 = 1; \tag{12}$$
$$SN_1^{-1} = 1,\ SN_1^0 = 1,\ SN_1^{+1} = 1;$$
$$SN_2^{-2} = \sqrt{\tfrac{1}{12}},\ SN_2^{-1} = \sqrt{\tfrac{1}{3}},\ SN_2^0 = 1,\ SN_2^{+1} = \sqrt{\tfrac{1}{3}},\ SN_2^{+2} = \sqrt{\tfrac{1}{12}};$$
$$SN_3^{-3} = \sqrt{\tfrac{1}{360}},\ SN_3^{-2} = \sqrt{\tfrac{1}{60}},\ SN_3^{-1} = \sqrt{\tfrac{1}{6}},\ SN_3^0 = 1,\ SN_3^{+1} = \sqrt{\tfrac{1}{6}},\ SN_3^{+2} = \sqrt{\tfrac{1}{60}},\ SN_3^{+3} = \sqrt{\tfrac{1}{360}};$$